Complex - Analysis

(Defn) Complex number:- The numbers of the form $a+ib$ where a & b are Real numbers is known as a Complex number, a is known as the Real Part and b is imaginary Part. If $z = a+ib$.

Then the Conjugate of $z = \bar{z} = a+ib$

$|z| =$ modulus of $z = \sqrt{a^2+b^2}$

origin, $z = \tan^{-1}\frac{b}{a}$.

(Defn) Neighbourhood of a Point z_0:- The $\{z : |z-z_0| < \delta\}$ is known as then neighbourhood of z_0 of radius δ. Where δ is Small Positive numbers.

(Defn) Limit Point:- A Point z_0 is Said to be a limited Point or a limiting Point of a Sets of Points S in the argand Plane, if every neighbourhood of z_0 Contains a Point of S, other than z_0.

(Defn) Interior Points and boundary Points:- There are two types of limit Point is known as interior Point and boundary Points. A limit Point z_0 is set to be an interior Point, if there exist a neighbourhood of z_0 which Contains only Points S as well as Points out Side S.

(Defn) open and Closed Set:- If every Point of a Set S is an inner Point of S, then S is Said to be an open Set. A Set S is Said to be a Closed Set, if it Contains all of its limiting Points.

(Defn) Bounded Set and unbounded Set:- A Set S is Said to be bounded if there exists a finite +ve number K Such that $|z| \leq K$ for all $z \in S$. A Set is Said to be unbounded if it is not bounded.

(Defn) Function of Complex variable:- A Complex variable w is Said to be a function of the Complex variable z. if is each $z \in D$ (Some region) there Corresponds one or more (definite) value of w. In this Case we write $w = f(z)$.

(Defn) Jordan arc:- The Set of Points z determined by the equation $z = x(t) + iy(t)$, where $x(t)$ & $y(t)$ are real variable of t, where $\alpha \leq t \leq \beta$ is Known as a Continuous arc. A Point z_1 is Known as a multiple Point of the arc if the equation $z_1 = x(t) + iy(t)$ is Satisfied by only two valued t then the a Point Said to be a double Point. A Jordan arc is a Continuous arc without multiple Points.

(Defn) Simple closed Jordan curve:- If the two end Points of the arc $z = x(t) + iy(t)$ Corresponding to the terminal values α & β of t, one Sides then the arc is Known as a Simple closed Jordan curve.

(Defn) Limit of a Complex valued function:- Let $w = f(z)$ is a Complex function, we define $\lim_{z \to z_0} f(z) = l$. If therefore a given ϵ, there exist Some δ. $|f(z) - l| < \epsilon$, $\forall$ z Satisfying $|z - z_0| < \delta$

Here, $z \to z_0$ along any curve.

(Defn) **Uniform Continuity:-** A function $f(z)$ is said to be uniformaly continuous in the domain D, if for a given $\epsilon > 0$, there exist a +ve number δ depending upon ϵ only such that for a any pair of points $z_1, z_2 \in D$.

$|f(z_1) - f(z_2)| < \epsilon$ Provided $|z_1 - z_2| < \delta$.

(Defn) **Differentiablity function:-** Let $w = f(z)$ be a ~~Point~~ function of z define in a domain D. $f(z)$ is said to differentiable at $z = a$. if

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(a+\Delta z) - f(a)}{\Delta z} \text{ exist,}$$

i.e. limit is finite and unique if this limit exists then we denote it by $f'(a)$ and called it the derivatives of $f(z)$ at $z = 0$.

QN→ Every differentiable function is continuous but the converse need not be true.

or, QN→ Prove that the function $|z|^2$ is continuous every where but nowhere differentiable except at the origin.

Proof:- Let $f(z)$ be differentiable at a Point $z = a$ in a domain D.

Hence, $$\lim_{\Delta z \to 0} \frac{f(a+\Delta z) - f(a)}{\Delta z} \text{ exist.}$$

In this case, we know that

$$\lim_{\Delta z \to 0} \frac{f(a+\Delta z) - f(a)}{\Delta z} = f'(a)$$

$$\therefore \lim_{\Delta z \to 0} \{f(a+\Delta z) - f(a)\} = \lim_{\Delta z \to 0} f'(a)\Delta z$$

$$= f'(a) \lim_{\Delta z \to 0} \Delta z$$

$$= f'(a) \times 0 = 0$$

$$\therefore \lim_{\Delta z \to 0} f(a+\Delta z) - \lim_{\Delta z \to 0} f(a) = 0$$

i.e. $\lim_{\Delta z \to 0} f(a+\Delta z) = f(a)$.

Hence, $f(z)$ is continuous at $z=a$

For the converse we consider the following example,

Let $f(z) = |z|^2$

Clearly, $f(z) = |z|^2 = x^2 + y^2$.

Hence, $f(z)$ is continuous every where

Since, x^2 & y^2 are continuous every where,

Now, $\lim_{\Delta z \to 0} \dfrac{f(a+\Delta z) - f(a)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{|a+\Delta z|^2 - |a|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(a+\Delta z)(\bar{a}+\overline{\Delta z}) - a\bar{a}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{a}\,\Delta z + a\,\overline{\Delta z} + \Delta z\,\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\bar{a} + a\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z}\right)$$

$$= \lim_{\Delta z \to 0} \left(\bar{a} + a\frac{\overline{\Delta z}}{\Delta z}\right) \quad [\because \overline{\Delta z} \to 0 \text{ as } \Delta z \to 0]$$

at $a=0$, clearly, the limits becomes zero, i.e. $f(z)$ is differentiable at $z=0$ if $a \neq 0$.

let $\Delta z = \rho(\cos\phi + i\sin\phi)$

$$\therefore \overline{\Delta z} = \rho(\cos\phi - i\sin\phi)$$

$$\therefore \underset{\Delta z \to 0}{Lt} \frac{f(a+\Delta z) - f(a)}{\Delta z}$$

$$= \underset{\Delta z \to 0}{Lt} \frac{\bar{a} + a\,\rho(\cos\phi - i\sin\phi)}{\rho(\cos\phi + i\sin\phi)}$$

$$= \underset{\Delta z \to 0}{Lt} \{\bar{a} + a\cos 2\phi - i\sin 2\phi\}$$

$$= \bar{a} + a(\cos 2\phi - i\sin 2\phi)$$

Which Cannot be uniaue.

Sence, its depend upon ϕ.

Hence, $f(z)$ is not dibberentiable at

$z = a$ not eaual to 0.

However, it is Continuous at $z = a \neq 0$.